Lagrangian Formulation of a Solution to the Cosmological Constant Problem

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Abstract

A covariant Lagrangian formulation of a solution to the cosmological constant problem, based on vizualising the fluctuations of the vacuum energy as a non-equilibrium process with stochastic behaviour, is presented. The variational principle yields equations of motion for the cosmological "constant" Λ, treated as a dynamical field, together with an equation for a Lagrange multiplier field ϕ, and the standard Einstein field equations with a variable cosmological constant term. A stochastic model of Λ yields a natural explanation for the smallness or zero value of the constant in the present epoch and its large value in an era of inflation in the early universe.

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A recent model for solving the cosmological constant problem has been proposed [1], in which the vacuum energy is treated as a fluctuating environment with stochastic behaviour. In the following, we shall present a covariant formulation based on a Lagrangian density, which yields classical equations incorporating Einstein's gravitational field equations, upon which a stochastic treatment using a Wiener process can be developed.

Although the model uses methods of critical phenomena and non-equilibrium statistical mechanics to model the vacuum energy, it can be considered as a phenomenological description of the kinds of behaviour that could be expected in a more fundamental quantum gravity theory. It is generally agreed that the cosmological constant problem cannot be solved within the context of a purely classical theory of gravity. However, no satisfactory quantum gravity theory has been formulated, so it is hoped that our model can shed light on the solution to the problem without the full apparatus of such a theory.

The Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_\Lambda + \mathcal{L}_M, \tag{1}$$

where

$$\mathcal{L}_R = \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}, \tag{2a}$$

$$\mathcal{L}_\Lambda = -2\sqrt{-g}[\Lambda + (\Lambda_{,\mu} u^\mu - \alpha\Lambda + \Lambda^2)\phi], \tag{2b}$$

and $\Lambda = \Lambda(x)$ is the variable cosmological "constant", treated as a dynamical field, ϕ is a Lagrange multiplier field, $u^\mu = dx^\mu/d\tau$ is an observer's four-velocity along a world line in spacetime, α is a constant and $\mathcal{L}_M$ is a matter Lagrangian density. A variation of $\mathcal{L}$ with respect to ϕ and Λ yields the equations of motion:

$$\Lambda_{,\mu} u^\mu - \alpha\Lambda + \Lambda^2 = 0, \tag{3a}$$

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}\, u^\mu \phi)_{,\mu} + (\alpha - 2\Lambda)\phi - 1 = 0. \tag{3b}$$

Varying $\mathcal{L}$ with respect to $g^{\mu\nu}$ and using (3a) gives

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4}$$

The kinematical variable u^μ is the four-velocity of a fluid element along a world line in spacetime, associated with a fluid with density ρ and pressure p, so we do not vary $\mathcal{L}_\Lambda$ with respect to u^μ.

The cosmological constant enters through the vacuum energy density:

$$T_{V\mu\nu} = -\rho_V g_{\mu\nu} = -\frac{\Lambda_V}{8\pi G} g_{\mu\nu}. \tag{5}$$

Today, Λ has the small value, $\Lambda < 10^{-46}$ GeV^4, whereas generic inflation models require that Λ has a relatively large value during the inflationary epoch. This is the source of the cosmological constant problem.

The line element in the Friedmann-Robertson-Walker model is

$$d\tau^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right\}, \tag{6}$$

where $k = -1, 0, +1$ and we have used comoving coordinates with $u^\mu = (0, 0, 0, 1)$. Then, Eqs.(3a), (3b) and (4) become

$$\dot{\Lambda} = \alpha\Lambda - \Lambda^2, \tag{7a}$$

$$\dot{\phi} + \frac{3\dot{R}}{R}\phi + (\alpha - 2\Lambda)\phi - 1 = 0, \tag{7b}$$

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho_M}{3} + \frac{\Lambda}{3} - \frac{k}{R^2}, \tag{7c}$$

where $\dot{\Lambda} = d\Lambda/dt$, ρ_M denotes the mass density, H is the Hubble constant, and in the following H_0 and t_0 denote the present values of H and t, respectively. We define

$$\Omega_{\text{tot}} \equiv \Omega_M + \Omega_\Lambda = 1 - \Omega_k, \tag{8}$$

where $\Omega = 8\pi G\rho/3H^2$.

We shall treat the vacuum energy as a fluctuating environment and consider Λ as a variable characterizing the state of this system. The parameter α in Eq.(7a) corresponds to the difference between the growth and decline of particle-antiparticle annihilation in the vacuum, while the second term is a self-restriction term which limits the growth of Λ.

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In ref.(1), we considered the situation in which the vacuum fluctuations are rapid compared with $\tau_{\text{macro}} = \alpha^{-1}$, which defines the macroscopic scale of time evolution. We assumed that the parameter α can be written as $\alpha_t = \alpha + \sigma\xi_t$, in which α is the average value, ξ_t is Gaussian noise and σ measures the intensity of the vacuum fluctuations. Let us write Eq.(7a) as

$$d\Lambda_t = (\alpha\Lambda_t - \Lambda_t^2)dt + \sigma\Lambda_t dW_t = f(\Lambda_t) + \sigma g(\Lambda_t)dW_t, \tag{9}$$

where dW_t is a Wiener process. The probability density $p(x,t)$, which describes the Λ distribution, satisfies the Fokker-Planck equation:

$$\partial_t p(x,t) = -\partial_x[(\alpha x - x^2)p(x,t)] + \frac{\sigma^2}{2}\partial_{xx}(x^2 p(x,t)). \tag{10}$$

The diffusion process is restricted to the positive real half line and 0 and ∞ are intrinsic boundaries, because $g(0) = 0$ and $f(\infty) = -\infty$. The probability of the diffusion process reaching infinity as $t \to \infty$ is zero, since infinity is a natural boundary. Moreover, zero is a natural boundary if $\alpha > \sigma^2/2$, so neither boundary is accessible and no boundary conditions need be imposed on the Fokker-Planck equation. For $\alpha < \sigma^2/2$, it can be shown that zero is an *attracting* boundary.

The stationary-state solution for the probability density, $p_s(x)$, of Eq.(10) is given by [2]

$$p_s(x) = Nx^{(2\alpha/\sigma^2)-2}\exp\left(-\frac{2x}{\sigma^2}\right). \tag{11}$$

The normalization constant N is

$$N^{-1} = \left[\left(\frac{2}{\sigma^2}\right)^{2(\alpha/\sigma^2)-1}\right]^{-1}\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right), \tag{12}$$

where Γ denotes the Γ-function. If $p(x,t)$ is integrable between 0 and ∞, then a stationary state solution exists when $\alpha > \sigma^2/2$. If it does not exist, then the probability density will be concentrated at zero, i.e., $p(\Lambda) = \delta(\Lambda)$ for $\alpha < \sigma^2/2$.

For $0 < \alpha < \sigma^2/2$, the vacuum fluctuations dominate over the growth or decline of Λ, although the value zero is still the most probable value for Λ, since the distribution function

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has a vertical slope at $\Lambda = 0$. Because we are using a continuous variable, Λ never reaches the boundary zero in a finite time.

When $\alpha > \sigma^2/2$, the growth of Λ dominates the influence of the vacuum fluctuations, and in the neighborhood of zero the probability of $\Lambda = 0$ drops to zero. For the stochastic model there are two transition points described by different order parameters. At $\alpha = \sigma^2/2$ real growth of Λ becomes possible corresponding to a change from a degenerate random variable for steady-state behavior to a stochastic variable; the boundary at $\alpha = 0$ switches from attracting to natural. Secondly, there is the transition point $\alpha = \sigma^2$ which corresponds to a qualitative change in the stochastic variable Λ with no change in the nature of the boundary. The probability of $\Lambda = 0$ drops abruptly to zero.

The following scenario can be deduced from our model. In the inflation era, the intensity of vacuum fluctuations is large and $\alpha > \sigma^2$, causing a second-order phase transition and a maximum in Λ not near zero. This corresponds to the large vacuum energy needed to drive inflation [3]. As the universe expands the intensity of vacuum fluctuations decreases and for $0 < \alpha < \sigma^2/2$ or $\sigma^2/2 < \alpha < \sigma^2$ the probability density is largest when Λ is non-vanishing and small, which can lead to a current value of Λ_0 that can be used to fit the observational data. If the stationary probability density p_s does not exist for $\alpha < \sigma^2/2$, then $\Lambda = 0$ is a stationary point; the drift and diffusion vanish simultaneously for $\Lambda = 0$ and $p(\Lambda) = \delta(\Lambda)$. This corresponds to the case when Λ is vanishingly small.

Thus, our model provides a natural explanation, in terms of non-equilibrium stochastic processes in an expanding universe, for the behavior of Λ required to fit observational data and still be consistent with inflationary models.

Ongoing searches [4] for Type Ia supernovae show that $\Omega_\Lambda < 0.47$ (at 95% confidence level for spatially flat Λ models). Moreover, for $\Lambda \neq 0$ a larger fraction of QSOs would be gravitationally lensed and QSO surveys give $\Omega_\Lambda \leq 0.7$ [5]. Cold dark matter models (CDM) for large scale stucture formation, which include a cosmological constant, yield a better fit to the shape of the observed spectrum of galaxy clustering than does the standard $\Omega_M = 1$ CDM models, using $h \equiv H_0/(100\mathrm{km/sec/Mpc}) = 0.7$, $\Omega_\Lambda = 0.6$ and a baryon density with

$\Omega_B = 0.0255$, consistent with primordial nucleosynthesis [6]. However, the amplitude for the Λ CDM models is too high compared to the data, a problem that persists at all scales.

The problem of the age of the universe is also alleviated in Λ models. An analysis of the cosmological data showed that for $\Omega_\Lambda = 0.65 \pm 0.1, \Omega_M = 1 - \Omega_\Lambda$ and a small tilt: $0.8 < n < 1.2$, models exist which are consistent with the available data and an inflationary spatially flat universe [7].

Models based on Λ treated as a scalar dynamical field [8] have been found to partially resolve observational problems. The observations of gravitationally lensed QSOs yield a less restrictive upper bound on $H_0 t_0$ in these models [9]. They may also provide a solution to the size of the amplitude problem, since although the shape of the spectrum is the same as that of the Λ CDM model with $\Omega_\Lambda = 0.6$, the dynamical Λ model yields a lower amplitude and therefore gives a better fit to the galaxy clustering data [9].

REFERENCES

[1] J. W. Moffat, astro-ph/9606071 (1996).

[2] W. Horsthemke and R. Lefever, *Noise-Induced Transitions, Theory and Applications in Physics, Chemistry, and Biology*, Springer-Verlag, Berlin, Heidelberg, New York, 1984.

[3] A. D. Linde, *Particle Physics and Inflationary Cosmology*, Gordon and Breach, New York, 1990.

[4] S. Perlmutter *et al.,* astro-ph/9602122 (1996).

[5] C. S. Kochanek, astro-ph/9510077, Astrophys. J., in press (1996); Astrophys. J. **419**, 12 (1993).

[6] J. A. Peacock and S. J. Dodds, MNRAS, **267**, 1020 (1994).

[7] J. P. Ostricker and P. J. Steinhardt, Nature **377**, 600 (1995); L. Krauss and M. S. Turner, Gen. Rel. Grav. **27**, 1137 (1995).

[8] J. W. Moffat, Phys. Lett. B **357**, 526 (1995). This article contains further references to dynamical scalar Λ models. See also, J. A. Frieman *et al.,* Phys. Rev. Lett. **75**, 2077 (1995).

[9] K. Coble, S. Dodelson, and J. A. Frieman, astro-ph/9608122 (1996).